

A Bernstein Inequality For Spatial Lattice Processes*

Eduardo Valenzuela-Domínguez[†]

Johannes T. N. Krebs^{‡§}

Jürgen E. Franke[¶]

February 8, 2017

Abstract

In this article we present a Bernstein inequality for sums of random variables which are defined on a spatial lattice structure. The inequality can be used to derive concentration inequalities. It can be useful to obtain consistency properties for nonparametric estimators of conditional expectation functions.

Keywords: Asymptotic inference; Asymptotic inequalities; Bernstein inequality; Concentration inequality; Nonparametric statistics; Spatial Lattice Processes; Strong mixing

MSC 2010: Primary: 62G20; 62M40; 37A25; Secondary: 62G05; 62G09

1 Introduction

Inequalities of the Bernstein type are a major tool for the asymptotic analysis in probability theory and statistics. The original inequality published by Bernstein (1927) considers the case $\mathbb{P}(|S_n| > \varepsilon)$, where $S_n = \sum_{k=1}^n Z_k$ for real-valued zero-mean random variables Z_1, \dots, Z_n which are independent and identically distributed and bounded. A short proof is given in Bosq (2012) together with a demonstration how Hoeffding's inequality (Hoeffding (1963)) can be concluded too. A version for independent multivariate random variables is given by Ahmad and Amezziane (2013).

Starting with Collomb's and Carbon's inequalities (Collomb (1984) and Carbon (1983)), during the last thirty years there have been derived various generalizations of Bernstein's inequality to stochastic processes $\{Z(t) : t \in \mathbb{Z}\}$ under the assumption of weak dependence (Bryc and Dembo (1996) and Merlevède et al. (2009)). The corresponding definitions of dependence and their interaction properties can be found in Doukhan (1994) and in Bradley (2005).

Furthermore, there are inequalities of the Bernstein-type which are tailored to special mathematical questions: Arcones (1995) develop Bernstein-type inequalities for U -statistics. Krebs (2017) gives a Bernstein inequality for strong mixing random fields which are defined on exponentially growing graphs.

Bernstein inequalities often find their applications when deriving large deviation results or (uniform) asymptotic consistency statements in nonparametric regression and density estimation: Valenzuela-Domínguez (1995) considers nonlinear function estimation on random random fields under mixing conditions. Such statistical procedures are also widely used in image analysis, where the image is modeled as a given function on part of the integer lattice \mathbb{Z}^2 contaminated by additive noise. Frequently, the noise is assumed to consist of independent and identically distributed random variables, but this assumption is not always realistic, compare e.g. Daul et al. (1998). A more general noise model is provided by stationary stochastic processes, e.g. by Markov random fields. For such processes, functions like conditional probability densities or conditional expectations of an observation given data in a neighborhood may also be estimated by nonparametric procedures Tran (1990). For investigating the asymptotic properties of those estimation procedures a Bernstein

*This research was partially supported by the Fraunhofer ITWM, 67663 Kaiserslautern, Germany and by the RTG 1932 Stochastic Models for Innovations in the Engineering Sciences funded by the German Research Foundation (DFG).

[†]Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile, email: eduardo.valenzuela@usm.cl

[‡]Department of Mathematics, University of Kaiserslautern, 67653 Kaiserslautern, Germany, email: krebs@mathematik.uni-kl.de

[§]Corresponding author

[¶]Department of Mathematics, University of Kaiserslautern, 67653 Kaiserslautern, Germany, email: franke@mathematik.uni-kl.de

inequality for spatial stochastic processes on an integer lattice is needed. For continuous-parameter processes on \mathbb{R}^2 , such a result has been derived in Bertail et al. (2000). Here, we provide a Bernstein inequality for stochastic processes on \mathbb{Z}^N under rather general conditions, e.g., assuming only α -mixing which is a rather weak type of mixing condition. To allow for other applications, e.g., to spatial-temporal processes used in modeling environmental data like precipitation or pollution, we do not restrict ourselves to the plane but consider integer lattices in arbitrary dimensions.

This paper is organized as follows: we give the main definitions and notation in Section 2. In Section 3 we present the Bernstein inequality for random fields on a lattice \mathbb{Z}^N and further concentration inequalities, it is the main part of this article.

2 Definitions and Notation

In this section we give the mathematical definitions and notation which we shall use to derive the results. Let there be given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $N \in \mathbb{N}$ be a natural number. A real-valued random field Z which is indexed by \mathbb{Z}^N is a collection of random variables $\{Z(s) : s \in \mathbb{Z}^N\}$. We denote by d_∞ the metric on the lattice \mathbb{Z}^N which is induced by the Euclidean- ∞ -norm, i.e., $d_\infty(s, t) = \max\{|s_i - t_i| : 1 \leq i \leq N\}$ for $s, t \in \mathbb{Z}^N$. We denote for two subsets $I, J \subseteq \mathbb{Z}^N$ their distance by

$$d_\infty(I, J) = \inf\{d_\infty(s, t) : s \in I, t \in J\}.$$

Furthermore, we write $s \leq t$ if and only if $s_i \leq t_i$ for $i = 1, \dots, N$.

The α -mixing coefficient describes the dependence between random variables, it is introduced by Rosenblatt (1956):

Definition 2.1 (α -mixing coefficient). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Given two sub- σ -algebras \mathcal{F} and \mathcal{G} of \mathcal{A} , the α -mixing coefficient is defined by

$$\alpha(\mathcal{F}, \mathcal{G}) := \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}, B \in \mathcal{G}\}.$$

Note that $\alpha(\mathcal{F}, \mathcal{G}) \leq 1/4$, compare Bradley (2005). If X and Y are two random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, then $\alpha(X, Y)$ is the mixing coefficient $\alpha(\sigma(X), \sigma(Y))$. Furthermore, for a random field $\{Z(s) : s \in \mathbb{Z}^N\}$ and a subset $I \subseteq \mathbb{Z}^N$, denote by $\mathcal{F}(I) := \sigma(Z(s) : s \in I)$ the σ -algebra generated by the $Z(s)$ in I . The α -mixing coefficient of the random field Z is then for $k \in \mathbb{N}$ defined as

$$\alpha(k) := \sup_{\substack{I, J \subseteq \mathbb{Z}^N, \\ d_\infty(I, J) \geq k}} \sup_{\substack{A \in \mathcal{F}(I), \\ B \in \mathcal{F}(J)}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \quad (2.1)$$

The random field Z is said to be strong (spatial) mixing (or α -mixing) if $\alpha(k) \rightarrow 0$ ($k \rightarrow \infty$).

We write $e_N = (1, \dots, 1)$ for the element in \mathbb{Z}^N which only contains ones. Let $n = (n_1, \dots, n_N) \in \mathbb{N}^N$, then we write I_n for the N -dimensional cube on the lattice which is spanned by e_N and n , i.e., $I_n = \{k \in \mathbb{Z}^N : e_N \leq k \leq n\}$.

3 Exponential inequalities for α -mixing processes on N -dimensional lattices

Theorem 3.1 (Bernstein inequality). Let $Z := \{Z(s) : s \in \mathbb{Z}^N\}$ be a real-valued random field defined on the N -dimensional lattice \mathbb{Z}^N . Each $Z(s)$ is bounded by a uniform constant B , has expectation zero and the variance of $Z(s)$ is uniformly bounded by σ^2 . Let Z be strong mixing with mixing coefficients $\{\alpha(k) : k \in \mathbb{N}\}$. Set $\bar{\alpha}_k := \sum_{u=1}^k u^{N-1} \alpha(u)$. Let $P(n) = (P_1(n_1), \dots, P_N(n_N))$ and $Q(n) = (Q_1(n_1), \dots, Q_N(n_N))$ be arbitrary non-decreasing sequences in \mathbb{N}^N which are indexed by $n \in \mathbb{N}^N$ and which satisfy for each $1 \leq k \leq N$

$$1 \leq Q_k(n_k) \leq P_k(n_k) < Q_k(n_k) + P_k(n_k) < n_k. \quad (3.1)$$

Furthermore, let $\mathbf{n} := |I_n| = n_1 \cdot \dots \cdot n_N$, $\mathbf{P} := P_1(n_1) \cdot \dots \cdot P_N(n_N)$ and $\underline{q} := \min \{Q_1(n_1), \dots, Q_N(n_N)\}$ as well as $\bar{p} := \max \{P_1(n_1), \dots, P_N(n_N)\}$. Then for all $\varepsilon > 0$ and $\beta > 0$ such that $2^{N+1} B \mathbf{P} e \beta < 1$

$$\mathbb{P} \left(\left| \sum_{s \in I_n} Z(s) \right| > \varepsilon \right) \leq 2 \exp \left\{ 12 \sqrt{e} 2^N \frac{\mathbf{n}}{\mathbf{P}} \alpha(\underline{q})^{\mathbf{P}} / [\mathbf{n}(2^N+1)] \right\} \cdot \exp \left\{ -\beta \varepsilon + 2^{3N} \beta^2 e (\sigma^2 + 12 B^2 \gamma \bar{\alpha} \bar{p}) \mathbf{n} \right\}. \quad (3.2)$$

Proof. We write $S_n = \sum_{s \in I_n} Z(s)$ for $n \in \mathbb{N}^N$. To exploit the mixing property we want to decompose the sum S_n into different parts which consist of sums over groups of the $Z(s)$. Using the mixing condition, most of these subsums are only weakly dependent. To simplify notation, we write

$$P \equiv P(n) \equiv (P_1, \dots, P_N), \quad Q \equiv Q(n) \equiv (Q_1, \dots, Q_N)$$

keeping the dependence on n in mind. We choose a corresponding sequence $R \equiv R(n) \equiv (R_1, \dots, R_N)$ such that

$$(R_k - 1)(P_k + Q_k) < n_k \leq R_k(P_k + Q_k) =: n_k^* \text{ for each } k = 1, \dots, N. \quad (3.3)$$

For the k -th coordinate direction, we partition the summation index set $\{1, \dots, n_k^*\} \supseteq \{1, \dots, n_k\}$ into R_k subsets each consisting of two disjoint intervals of length P_k and Q_k resp. So, we have a union of $2R_k$ intervals half of them of length P_k , the other half of length Q_k , covering the set $\{1, \dots, n_k\}$.

Combining the partitions in all N coordinate directions, we get a partition of the N -dimensional rectangle $I_{n^*} = \{s \in \mathbb{Z}^N; e_N \leq s \leq n^*\} \supseteq I_n$ into $\mathbf{R} = R_1 \cdot \dots \cdot R_N$ blocks containing $(P_1 + Q_1) \cdot \dots \cdot (P_N + Q_N)$ points of the N -dimensional integer lattice each. Within each block, there are 2^N smaller subsets, which are N -dimensional rectangles with all edges of length either P_k or Q_k , $k = 1, \dots, N$. Write $I(l, u)$ for the l -th subset in the u -th block, $l = 1, \dots, 2^N$ and $u = 1, \dots, \mathbf{R}$. Note that the diameter w.r.t. d_∞ of the rectangular set $I(l, u)$ is bounded by \bar{p} , since

$$\text{diam}\{I(l, u)\} = \max\{d_\infty(s, t), s, t \in I(l, u)\} \leq \max\{P_1, \dots, P_N\} = \bar{p}. \quad (3.4)$$

Its cardinality is at most $\text{card}\{I(l, u)\} \leq \prod_{k=1}^N \max\{P_k, Q_k\} = \prod_{k=1}^N P_k = \mathbf{P}$ (cf. (3.1)). Now we can partition the sum $S_n = \sum_{s \in I_n} Z(s)$ as follows

$$S_n = \sum_{l=1}^{2^N} \sum_{u=1}^{\mathbf{R}} \sum_{s \in I(l, u)} Z(s) = \sum_{l=1}^{2^N} \sum_{u=1}^{\mathbf{R}} S(l, u) = \sum_{l=1}^{2^N} T(l, \mathbf{R})$$

with $S(l, u) = \sum_{s \in I(l, u)} Z(s)$ and $T(l, r) = \sum_{u=1}^r S(l, u)$, for $r = 1, \dots, \mathbf{R}$. We have the recursive property

$$T(l, r) = T(l, r-1) + S(l, r) \text{ and } T(l, 0) = 0. \quad (3.5)$$

Now we can apply this decomposition to the exponential $e^{\beta S_n}$ as follows

$$\mathbb{E} [e^{\beta S_n}] = \mathbb{E} [e^{\beta \sum_{l=1}^{2^N} T(l, \mathbf{R})}] = \mathbb{E} \left[\prod_{l=1}^{2^N} e^{\beta T(l, \mathbf{R})} \right] \leq \mathbb{E} \left[2^{-N} \sum_{l=1}^{2^N} e^{2^N \beta T(l, \mathbf{R})} \right] \quad (3.6)$$

where we have used the well-known inequality between geometric and arithmetic mean. Setting $\delta = 2^N \beta$ we have $\mathbb{E} [e^{\beta S_n}] \leq 2^{-N} \sum_{l=1}^{2^N} \mathbb{E} [e^{\delta T(l, \mathbf{R})}]$. Now, we study $\mathbb{E} [e^{\delta T(l, r)}]$ for $l = 1, \dots, 2^N$ and $r = 1, \dots, \mathbf{R}$. By (3.5)

$$\begin{aligned} \mathbb{E} [e^{\delta T(l, r)}] &= \mathbb{E} [e^{\delta T(l, r-1)} e^{\delta S(l, r)}] \\ &\leq \left| \mathbb{E} [e^{\delta T(l, r-1)} e^{\delta S(l, r)}] - \mathbb{E} [e^{\delta T(l, r-1)}] \mathbb{E} [e^{\delta S(l, r)}] \right| + \left| \mathbb{E} [e^{\delta T(l, r-1)}] \mathbb{E} [e^{\delta S(l, r)}] \right|. \end{aligned}$$

But $T(l, r-1)$ is $\mathcal{F}(I(l, 1) \cup \dots \cup I(l, r-1)) =: \mathcal{F}(J(l, r-1))$ -measurable and $S(l, r)$ is $\mathcal{F}(I(l, r))$ -measurable, this

implies that $e^{\delta T(l, r-1)}$ is $\mathcal{F}(J(l, r-1))$ -measurable and $e^{\delta S(l, r)}$ is $\mathcal{F}(I(l, r))$ -measurable. Since $Z(s)$ is bounded and the minimal distance between the sets $J(l, r-1)$ and $I(l, r)$ is $d_\infty(J(l, r-1), I(l, r)) \geq \min\{Q_1, \dots, Q_N\} = \underline{q}$, we can apply Davydov's inequality (compare A.1) as follows

$$\left| \mathbb{E} \left[e^{\delta T(l, r-1)} e^{\delta S(l, r)} \right] - \mathbb{E} \left[e^{\delta T(l, r-1)} \right] \mathbb{E} \left[e^{\delta S(l, r)} \right] \right| \leq 12\alpha(\underline{q})^{1/a} \|e^{\delta S(l, r)}\|_\infty \|e^{\delta T(l, r-1)}\|_b$$

with $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1$, therefore

$$\mathbb{E} \left[e^{\delta T(l, r)} \right] \leq 12\alpha(\underline{q})^{1/a} \|e^{\delta S(l, r)}\|_\infty \|e^{\delta T(l, r-1)}\|_b + \mathbb{E} \left[e^{\delta T(l, r-1)} \right] \mathbb{E} \left[e^{\delta S(l, r)} \right]. \quad (3.7)$$

As $|S(l, r)| \leq \sum_{s \in I(l, r)} |Z(s)| \leq B\mathbf{P}$ and choosing

$$0 < \beta \leq \frac{1}{2^{N+1}B\mathbf{P}e}, \text{ i.e., } 0 < \delta \leq \frac{1}{2B\mathbf{P}e}$$

we have $\delta S(l, r) \leq 1/(2e)$ and for all D such that $0 \leq D \leq e$

$$|\delta DS(l, r)| \leq \frac{1}{2} \quad (3.8)$$

which implies

$$\left\| e^{\delta DS(l, r)} \right\|_\infty \leq \sqrt{e}. \quad (3.9)$$

Using (3.8), we have $e^{\delta DS(l, r)} \leq 1 + \delta DS(l, r) + (\delta DS(l, r))^2$. Next, we take expectations of this inequality and use that the $Z(s)$ have expectation zero as well as that the inequality $1 + x \leq \exp x$ is true for all $x \geq 0$. We obtain

$$\mathbb{E} \left[e^{\delta DS(l, r)} \right] \leq 1 + \delta^2 D^2 \mathbb{E} [S(l, r)^2] \leq e^{\delta^2 D^2 \mathbb{E} [S(l, r)^2]}. \quad (3.10)$$

Now we have to evaluate $E[S(l, r)^2]$:

$$\mathbb{E} [S(l, r)^2] = \mathbb{E} \left[\left(\sum_{s \in I(l, r)} Z(s) \right)^2 \right] = \sum_{s \in I(l, r)} \mathbb{E} [Z(s)^2] + \sum_{s \in I(l, r)} \sum_{t \in I(l, r), t \neq s} \mathbb{E} [Z(s)Z(t)]$$

We know that $|Z(s)| \leq B$, so $|\mathbb{E} [Z(s)Z(t)]| \leq 12B^2\alpha(d_\infty(s, t))$ and using $\mathbb{E} [Z(s)^2] \leq \sigma^2 < \infty$, we have

$$\mathbb{E} [S(l, r)^2] \leq \sigma^2 \mathbf{P} + 12B^2 \sum_{s \in I(l, r)} \sum_{t \in I(l, r), t \neq s} \alpha(d_\infty(s, t))$$

In order to evaluate the double sum, note that if $s, t \in I(l, r)$, $s \neq t$, then by (3.4) $d_\infty(s, t)$ assumes values between 1 and \bar{p} , i.e., $1 \leq d_\infty(s, t) \leq \bar{p}$. Furthermore, for a general point $s \in \mathbb{Z}^N$ the cardinality of the set of points $t \in \mathbb{Z}^N$ whose distance to s is exactly u is $\text{card}\{t \in \mathbb{Z}^N : d_\infty(s, t) = u\} = (2u+1)^N - (2u-1)^N \leq \gamma u^{N-1}$ for $u \geq 1$, where γ is a constant which depends on the lattice dimension N . Thus, the double sum can be bounded as follows

$$\begin{aligned} \sum_{s \in I(l, r)} \sum_{t \in I(l, r), t \neq s} \alpha(d_\infty(s, t)) &\leq \sum_{s \in I(l, r)} \sum_{u=1}^{\bar{p}} \sum_{t \in \mathbb{Z}^N : d_\infty(s, t)=u} \alpha(u) \\ &\leq \sum_{s \in I(l, r)} \sum_{u=1}^{\bar{p}} \alpha(u) \{(2u+1)^N - (2u-1)^N\} \leq \gamma \mathbf{P} \sum_{u=1}^{\bar{p}} \alpha(u) u^N. \end{aligned}$$

So, we have

$$E[S(l, r)^2] \leq \sigma^2 \mathbf{P} + 12B^2 \gamma \bar{\alpha}_p \mathbf{P}. \quad (3.11)$$

From (3.10) we obtain $\mathbb{E} [e^{\delta D S(l, r)}] \leq \exp(\delta^2 D^2 (\sigma^2 \mathbf{P} + 12B^2 \gamma \bar{\alpha}_p \mathbf{P}))$. We set $V := \sigma^2 \mathbf{P} + 12B^2 \gamma \bar{\alpha}_p \mathbf{P}$ and $D = 1$. Thus, it follows from (3.7)

$$\mathbb{E} [e^{\delta T(l, r)}] \leq 12\alpha(\underline{q})^{1/a} \left\| e^{\delta S(l, r)} \right\|_{\infty} \left\| e^{\delta T(l, r-1)} \right\|_b + \mathbb{E} [e^{\delta T(l, r-1)}] e^{\delta^2 V}.$$

But by Hölder's inequality $\mathbb{E} [e^{\delta T(l, r-1)}] \leq \left\| e^{\delta T(l, r-1)} \right\|_b$, so we obtain

$$\mathbb{E} [e^{\delta T(l, r)}] \leq \left(e^{\delta^2 V} + 12\alpha(\underline{q})^{1/a} \left\| e^{\delta S(l, r)} \right\|_{\infty} \right) \left\| e^{\delta T(l, r-1)} \right\|_b. \quad (3.12)$$

Now let $a = 1 + r$ and $b = 1 + 1/r$ such that for all $i = 1, \dots, r$, we have

$$1 \leq b^{i-1} \leq \left(1 + \frac{1}{r} \right)^r \leq e. \quad (3.13)$$

Then we obtain successively as in deriving (3.12) the following inequalities for $r \geq 2$:

$$\begin{aligned} \left\| e^{\delta T(l, r-1)} \right\|_b &\leq \left(e^{\delta^2 b^2 V} + 12\alpha(\underline{q})^{1/a} \left\| e^{\delta b S(l, r-1)} \right\|_{\infty} \right)^{1/b} \left\| e^{\delta T(l, r-2)} \right\|_{b^2} \\ \left\| e^{\delta T(l, r-2)} \right\|_{b^2} &\leq \left(e^{\delta^2 b^4 V} + 12\alpha(\underline{q})^{1/a} \left\| e^{\delta b^2 S(l, r-2)} \right\|_{\infty} \right)^{1/b^2} \left\| e^{\delta T(l, r-3)} \right\|_{b^3} \\ &\vdots \\ \left\| e^{\delta T(l, 2)} \right\|_{b^{r-2}} &\leq \left(e^{\delta^2 b^{2(r-2)} V} + 12\alpha(\underline{q})^{1/a} \left\| e^{\delta b^{r-2} S(l, 2)} \right\|_{\infty} \right)^{1/b^{r-2}} \left\| e^{\delta T(l, 1)} \right\|_{b^{r-1}}. \end{aligned}$$

Substituting, we get:

$$\mathbb{E} [e^{\delta T(l, r)}] \leq \left[\prod_{i=1}^{r-1} (e^{\delta^2 b^{2(i-1)} V} + 12\alpha(\underline{q})^{1/a} \left\| e^{\delta b^{i-1} S(l, r-i+1)} \right\|_{\infty})^{1/b^{i-1}} \right] \mathbb{E} [e^{\delta b^{r-1} T(l, 1)}]^{1/b^{r-1}} \quad (3.14)$$

but $b^{i-1} \leq e$ for $i = 1, \dots, r$, such that $\left\| e^{\delta b^{i-1} S(l, r-i+1)} \right\|_{\infty} \leq \sqrt{e}$ by (3.9) and even further

$$\begin{aligned} (e^{\delta^2 b^{2(i-1)} V} + 12\alpha(\underline{q})^{1/a} \sqrt{e})^{1/b^{i-1}} &\leq e^{\delta^2 b^{i-1} V} (1 + 12\alpha(\underline{q})^{1/a} \sqrt{e})^{1/b^{i-1}} \\ &\leq \exp \left\{ \delta^2 b^{i-1} V + \frac{12\sqrt{e}\alpha(\underline{q})^{1/a}}{b^{i-1}} \right\} \\ &\leq \exp \{ \delta^2 b^{i-1} V \} \exp \{ 12\sqrt{e}\alpha(\underline{q})^{1/a} \} \end{aligned}$$

by (3.13). Therefore, again using (3.13)

$$\prod_{i=1}^{r-1} (e^{\delta^2 b^{2(i-1)} V} + 12\alpha(\underline{q})^{1/a} \sqrt{e})^{1/b^{i-1}} \leq \prod_{i=1}^{r-1} e^{12\sqrt{e}\alpha(\underline{q})^{1/a} + \delta^2 e V} = \exp \left\{ 12\sqrt{e}\alpha(\underline{q})^{1/a} (r-1) + \delta^2 e V (r-1) \right\}.$$

Since $b^{r-1} \leq e$ by (3.13), and using (3.10) and (3.11) we have

$$\begin{aligned} \left\| e^{\delta T(l, 1)} \right\|_{b^{r-1}} &\leq \left\| e^{\delta T(l, 1)} \right\|_e = \mathbb{E} [e^{\delta e T(l, 1)}]^{1/e} \leq \mathbb{E} [e^{\delta e S(l, 1)}]^{1/e} \\ &\leq (e^{\delta^2 e^2 \mathbb{E}[S(l, 1)^2]})^{1/e} \leq (e^{\delta^2 e^2 V})^{1/e} = \exp \{ \delta^2 e V \}. \end{aligned}$$

Combining these results, we get from (3.14) for $l = 1, \dots, 2^N$ and $r = 1, \dots, \mathbf{R}$ that

$$\mathbb{E} \left[e^{\delta T(l,r)} \right] \leq \exp \left\{ 12\sqrt{e}\alpha(\underline{q})^{1/a}(r-1) + \delta^2 e V r \right\}.$$

By (3.1), $P_k < P_k + Q_k < n_k$ for each $k = 1, \dots, N$ which implies by (3.3) that both $P_k < n_k$ and $R_k < 2n_k/P_k$. For $a = 1 + r, r = \mathbf{R}$, we therefore have the two relations $1 > \frac{1}{a} > \mathbf{P}/[(2^N + 1)\mathbf{n}]$ and $\mathbf{R} \leq 2^N \mathbf{n}/\mathbf{P}$. Hence, for the choice $r = \mathbf{R}$ we arrive at (using that $0 < \alpha(\underline{q}) \leq 1/4$)

$$\mathbb{E} \left[e^{\delta T(l,\mathbf{R})} \right] \leq \exp \left\{ 12\sqrt{e}\alpha(\underline{q})^{\mathbf{P}/[(2^N+1)\mathbf{n}]} \left(2^N \frac{\mathbf{n}}{\mathbf{P}} - 1 \right) + \delta^2 e V 2^N \frac{\mathbf{n}}{\mathbf{P}} \right\}.$$

Using $\delta = 2^N \beta$:

$$\mathbb{E} \left[e^{2^N \beta T(l,\mathbf{R})} \right] \leq \exp \left\{ 2^{3N} \beta^2 e V \frac{\mathbf{n}}{\mathbf{P}} + 12\sqrt{e}\alpha(\underline{q})^{\mathbf{P}/[(2^N+1)\mathbf{n}]} \left(2^N \frac{\mathbf{n}}{\mathbf{P}} - 1 \right) \right\}$$

Returning to (3.6) and using Markov's inequality, we have

$$\begin{aligned} \mathbb{P}(|S_n| > \varepsilon) &= \mathbb{P}(S_n > \varepsilon) + \mathbb{P}(-S_n > \varepsilon) = \mathbb{P}(e^{\beta S_n} > e^{\beta \varepsilon}) + \mathbb{P}(e^{-\beta S_n} > e^{\beta \varepsilon}) \\ &\leq e^{-\beta \varepsilon} \left\{ \mathbb{E} \left[e^{\beta S_n} \right] + \mathbb{E} \left[e^{-\beta S_n} \right] \right\} \end{aligned}$$

Now, if we change $Z(s)$ to $-Z(s)$, all results remain valid, therefore we have in (3.6)

$$\begin{aligned} \mathbb{P}(|S_n| > \varepsilon) &\leq e^{-\beta \varepsilon} \left\{ 2^{-N} \sum_{l=1}^{2^N} \left(\mathbb{E} \left[e^{2^N \beta T_{l,\mathbf{R}}} \right] + \mathbb{E} \left[e^{-2^N \beta T_{l,\mathbf{R}}} \right] \right) \right\} \\ &\leq 2e^{-\beta \varepsilon} \exp \left\{ 2^{3N} \beta^2 e V \frac{\mathbf{n}}{\mathbf{P}} + 12\sqrt{e}\alpha(\underline{q})^{\mathbf{P}/[(2^N+1)\mathbf{n}]} \left(2^N \frac{\mathbf{n}}{\mathbf{P}} - 1 \right) \right\}. \end{aligned}$$

Recalling the definition of V this immediately implies (3.2). \square

We can formulate the following extension of the above Bernstein inequality to unbounded random variables

Theorem 3.2. *Let $\{Z(s) : s \in I\}$ be a strong mixing random field with $\mathbb{E}[Z(s)] = 0$ and $\mathbb{E}[Z(s)^2] \leq \sigma^2 < \infty$. Furthermore, assume that the tail distribution is bounded uniformly in s by*

$$\mathbb{P}(|Z(s)| > z) \leq \kappa_0 \exp(-\kappa_1 z^\tau) \quad (3.15)$$

for $\kappa_0, \kappa_1, \tau > 0$. Then, for any $B > 0$, we have with the notation from Theorem 3.1 and for $\mathbf{n} = |I_n|$

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{s \in I_n} Z(s) \right| > \varepsilon \right) &\leq \frac{12}{\varepsilon \tau} \kappa_0 \kappa_1^{-1/\tau} \Gamma(\tau^{-1}, \kappa_1 B^\tau) \mathbf{n} + 2 \exp \left\{ 12\sqrt{e} 2^N \frac{\mathbf{n}}{\mathbf{P}} \alpha(\underline{q})^{\mathbf{P}/[\mathbf{n}(2^N+1)]} \right\} \\ &\quad \cdot \exp \left\{ -\frac{1}{3} \beta \varepsilon \right\} \cdot \exp \left\{ 2^{3N} \beta^2 e (\sigma^2 + 48B^2 \gamma \bar{\alpha}_{\overline{P}}) \mathbf{n} \right\} \end{aligned}$$

where Γ denotes the upper incomplete gamma function.

Proof. We split each $Z(s)$: choose an arbitrary bound $B > 0$ and define for $s \in \mathbb{Z}^N$

$$\begin{aligned} Z(s)^\# &:= Z(s) - \min(Z(s), B) \geq 0, \quad Z(s)^* := Z(s) - \max(Z(s), -B) \leq 0 \\ &\text{and } Z(s)^0 := \max(\min(Z(s), B), -B). \end{aligned}$$

Then, $Z(s) = Z(s)^\# + Z(s)^* + Z(s)^0$ and $0 = \mathbb{E}[Z(s)] = \mathbb{E}[Z(s)^\#] + \mathbb{E}[Z(s)^*] + \mathbb{E}[Z(s)^0]$. Thus,

$$\begin{aligned} \mathbb{P}\left(\left|\sum_{s \in I_n} Z(s)\right| > \varepsilon\right) &= \mathbb{P}\left(\left|\sum_{s \in I_n} Z(s) - \mathbb{E}[Z(s)]\right| > \varepsilon\right) \\ &\leq \mathbb{P}\left(\left|\sum_{s \in I_n} Z(s)^\# - \mathbb{E}[Z(s)^\#]\right| > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(\left|\sum_{s \in I_n} Z(s)^* - \mathbb{E}[Z(s)^*]\right| > \frac{\varepsilon}{3}\right) \\ &\quad + \mathbb{P}\left(\left|\sum_{s \in I_n} Z(s)^0 - \mathbb{E}[Z(s)^0]\right| > \frac{\varepsilon}{3}\right). \end{aligned} \quad (3.16)$$

We treat each term of (3.16) separately. We consider the first two terms. Using Chebyshev's inequality we obtain

$$\mathbb{P}\left(\left|\sum_{s \in I_n} Z(s)^\# - \mathbb{E}[Z(s)^\#]\right| > \frac{\varepsilon}{3}\right) \leq \frac{3}{\varepsilon} \mathbb{E}\left[\left|\sum_{s \in I_n} Z(s)^\# - \mathbb{E}[Z(s)^\#]\right|\right] \leq \frac{6|I_n|}{\varepsilon} \mathbb{E}[Z(s)^\#]. \quad (3.17)$$

Using the tail condition, we can estimate the expectation in (3.17) by

$$\begin{aligned} \mathbb{E}[Z(s)^\#] &= \int_0^\infty \mathbb{P}(Z(s)^\# > z) \, dz \\ &= \int_0^\infty \mathbb{P}((Z(s) - B)1_{\{Z(s) \geq B\}} > z) \, dz = \int_B^\infty \mathbb{P}(Z(s) > z) \, dz \\ &\leq \kappa_0 \int_B^\infty \exp(-\kappa_1 z^\tau) \, dz = \kappa_0 \int_{\kappa_1 B^\tau}^\infty \frac{1}{\tau} \left(\frac{1}{\kappa_1}\right)^{1/\tau} y^{\frac{1}{\tau}-1} e^{-y} \, dy = \frac{\kappa_0}{\tau} \left(\frac{1}{\kappa_1}\right)^{1/\tau} \Gamma\left(\frac{1}{\tau}, \kappa_1 B^\tau\right). \end{aligned}$$

Since $\sigma(Z(s)^0 : s \in I) \subseteq \sigma(Z(s) : s \in I)$ for any $I \subseteq \mathbb{Z}^N$, the mixing coefficient of the field $\{Z(s)^0 : s \in \mathbb{Z}^N\}$ can be estimated by those of $\{Z(s) : s \in \mathbb{Z}^N\}$. Furthermore, $\text{Var}(Z(s)^0) \leq \sigma^2$ and we can apply Theorem 3.1 to the third term of (3.16), using that $|Z(s)^0 - \mathbb{E}[Z(s)^0]| \leq 2B$. Hence,

$$\begin{aligned} &\mathbb{P}\left(\left|\sum_{s \in I_n} Z(s)^0 - \mathbb{E}[Z(s)^0]\right| > \frac{\varepsilon}{3}\right) \\ &\leq 2 \exp\left\{12\sqrt{e}2^N \frac{\mathbf{n}}{\mathbf{P}} \alpha(\underline{q})^{\mathbf{P}/[\mathbf{n}(2^N+1)]}\right\} \cdot \exp\left\{-\frac{\varepsilon}{3}\beta\right\} \cdot \exp\left\{2^{3N}\beta^2 e(\sigma^2 + 48B^2\gamma\bar{\alpha}_p) \mathbf{n}\right\}. \end{aligned}$$

This ends the proof. \square

We give a result which is an immediate consequence of Theorem 3.1:

Corollary 3.3. *Let the real valued random field Z have α -mixing coefficients which are exponentially decreasing, i.e., there are $c_0, c_1 \in \mathbb{R}_+$ such that $\alpha(k) \leq c_0 \exp(-c_1 k)$. The $Z(s)$ have expectation zero and are bounded by B . Let $n \in \mathbb{N}^N$ be such that both*

$$\min_{1 \leq i \leq N} n_i \geq e^2 \text{ and } \frac{\min\{n_i : i = 1, \dots, N\}}{\max\{n_i : i = 1, \dots, N\}} \geq C',$$

for a constant $C' > 0$. There are constants $A_1, A_2 \in \mathbb{R}_+$ which depend on the lattice dimension N , the constant C' and the bound on the mixing coefficients but not on $n \in \mathbb{N}^N$ and not on B such that for all $\varepsilon > 0$

$$\mathbb{P}\left(\left|\sum_{s \in I_n} Z(s)\right| > \varepsilon\right) \leq A_1 \exp\left(-A_2 \varepsilon B^{-1} \mathbf{n}^{-N/(N+1)} \left(\prod_{i=1}^N \log n_i\right)^{-1}\right).$$

Proof of Corollary 3.3. Define $P_i(n_i) := Q_i(n_i) := \lfloor n_i^{N/(N+1)} \log n_i \rfloor$ for $i = 1, \dots, N$. Furthermore, we denote the smallest coordinate of $n \in \mathbb{N}^N$ by $\underline{n} := \min_{1 \leq i \leq N} n_i$ and the largest coordinate by $\bar{n} = \max_{1 \leq i \leq N} n_i$. Note that $\bar{n} \rightarrow \infty$ implies that $\underline{n} \rightarrow \infty$. We consider the first factor of the RHS of (3.2) and show that under the stated conditions we have

$$\sup \left\{ \exp \left(12\sqrt{e} 2^N \frac{\mathbf{n}}{\mathbf{P}} \alpha(\underline{q})^{\mathbf{P}/[\mathbf{n}(2^N+1)]} \right) : n \in \mathbb{Z}^N, \underline{n} \geq e^2 \right\} < \infty. \quad (3.18)$$

By assumption we have that $\alpha(\underline{q}) \leq c_1 \exp(-c_2 \underline{q})$, for two constants $c_1, c_2 \in \mathbb{R}_{\geq 0}$ and $\underline{q} = \min_{1 \leq i \leq N} Q_i$. Therefore it suffices to show that

$$\log(\mathbf{n}/\mathbf{P}) - c_2/(2^N + 1) \underline{q} \mathbf{P}/\mathbf{n} \rightarrow -\infty \text{ as } \underline{n} \rightarrow \infty, \quad (3.19)$$

because a bounded minimum \underline{n} implies a bounded maximum as well, i.e., for an arbitrary bound $0 < S < \infty$ the set

$$\{n \in \mathbb{N}^N : \underline{n}/\bar{n} \geq C' \text{ and } e^2 \leq \underline{n} < S\}$$

is finite. Note that for $a, b \geq 2$, we have $ab \geq a + b$. Thus, for $\prod_{i=1}^N \log n_i \geq \sum_{i=1}^N \log n_i$ if \underline{n} is at least e^2 . We make the definition $\eta := N/(N+1)$. Let $\underline{n} \geq e^2$, then for any constant $c \in \mathbb{R}_+$

$$\begin{aligned} & \log \left(\left(\prod_{i=1}^N n_i \right)^{1-\eta} \left(\prod_{i=1}^N \log n_i \right)^{-1} \right) - c(\underline{n})^\eta \log \underline{n} \left(\prod_{i=1}^N n_i \right)^{\eta-1} \left(\prod_{i=1}^N \log n_i \right) \\ & \leq (N+1)^{-1} \sum_{i=1}^N \log n_i - c \frac{(\underline{n})^{\eta+(\eta-1)}}{(\bar{n})^{(N-1)(1-\eta)}} \left(\log \underline{n} \prod_{i=1}^N \log n_i \right) \\ & \leq (N+1)^{-1} \prod_{i=1}^N \log n_i - c \left(\frac{\underline{n}}{\bar{n}} \right)^{(N-1)/(N+1)} \left(\log \underline{n} \prod_{i=1}^N \log n_i \right) \\ & = \left((N+1)^{-1} - c \left(\frac{\underline{n}}{\bar{n}} \right)^{(N-1)/(N+1)} \log \underline{n} \right) \prod_{i=1}^N \log n_i \rightarrow -\infty \text{ as } \underline{n} \rightarrow \infty. \end{aligned}$$

This proves (3.19) and consequently, that (3.18) is finite. We come to the second term inside the second factor of (3.2). Define $\beta := (2^{N+2} e B \mathbf{P})^{-1}$ which fulfills the requirements of Theorem 3.1. Then,

$$\sup \{ 2^{3N} \beta^2 e(\sigma^2 + 12B^2 \gamma \bar{\alpha}_p) \mathbf{n} : n \in \mathbb{N}^N, \underline{n} \geq e^2 \} < \infty. \quad (3.20)$$

This proves that $\mathbb{P}(|\sum_{s \in I_n} Z(s)| > \varepsilon) \leq A \exp(-\varepsilon/(2^{N+2} e B \mathbf{P}))$ for a constant $A \in \mathbb{R}_+$. \square

A Appendix

Davydov's inequality relates the covariance of two random variables to the α -mixing coefficient:

Proposition A.1 (Davydov (1968)). *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\mathcal{G}, \mathcal{H} \subseteq \mathcal{A}$ be sub- σ -algebras. Denote by $\alpha := \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{G}, B \in \mathcal{H}\}$ the α -mixing coefficient of \mathcal{G} and \mathcal{H} . Let $p, q, r \geq 1$ be Hölder conjugate, i.e., $p^{-1} + q^{-1} + r^{-1} = 1$. Let ξ (resp. η) be in $L^p(\mathbb{P})$ and \mathcal{G} -measurable (resp. in $L^q(\mathbb{P})$ and \mathcal{H} -measurable). Then $|\text{Cov}(\xi, \eta)| \leq 12 \alpha^{1/r} \|\xi\|_{L^p(\mathbb{P})} \|\eta\|_{L^q(\mathbb{P})}$.*

References

I. A. Ahmad and M. Amezziane. Probability inequalities for bounded random vectors. *Statistics & Probability Letters*, 83(4):1136–1142, 2013.

- M. A. Arcones. A Bernstein-type inequality for U-statistics and U-processes. *Statistics & Probability Letters*, 22(3): 239–247, 1995.
- S. Bernstein. Sur l’extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes. *Mathematische Annalen*, 97(1):1–59, 1927.
- P. Bertail, D. N. Politis, and N. Rhomari. Subsampling continuous parameter random fields and a Bernstein inequality. *Statistics: A Journal of Theoretical and Applied Statistics*, 33(4):367–392, 2000.
- D. Bosq. *Linear processes in function spaces: theory and applications*, volume 149. Springer Science & Business Media, 2012.
- R. C. Bradley. Basic properties of strong mixing conditions. a survey and some open questions. *Probability Surveys*, 2(2):107–144, 2005.
- W. Bryc and A. Dembo. Large deviations and strong mixing. In *Annales de l’IHP Probabilités et statistiques*, volume 32, pages 549–569, 1996.
- M. Carbon. Inégalité de Bernstein pour les processus fortement mélangeants non nécessairement stationnaires. *C.R. Acad. Sc. Paris I*, 297:303–306, 1983.
- G. Collomb. Propriétés de convergence presque complète du prédicteur à noyau. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 66(3):441–460, 1984.
- C. Daul, R. Rösch, B. Claus, J. Grotepaß, U. Knaak, and R. Föhr. A fast image processing algorithm for quality control of woven textiles. In *Mustererkennung 1998*, pages 471–479. Springer, 1998.
- Y. A. Davydov. Convergence of distributions generated by stationary stochastic processes. *Theory of Probability & Its Applications*, 13(4):691–696, 1968.
- P. Doukhan. *Mixing, volume 85 of Lecture Notes in Statistics*. Springer-Verlag, New York, 1994.
- L. Györfi, M. Kohler, A. Krzyżak, and H. Walk. *A distribution-free theory of nonparametric regression*. Springer Berlin, New York, Heidelberg, 2002.
- W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.
- J. T. N. Krebs. A Bernstein inequality for exponentially growing graphs. *arXiv:1701.04188 [math, stat]*, Preprint, submitted, 2017a. URL <http://arxiv.org/abs/1701.04188>.
- F. Merlevède, M. Peligrad, and E. Rio. *Bernstein inequality and moderate deviations under strong mixing conditions*, volume Volume 5 of *Collections*, pages 273–292. Institute of Mathematical Statistics, Beachwood, Ohio, USA, 2009.
- M. Rosenblatt. A central limit theorem and a strong mixing condition. *Proceedings of the National Academy of Sciences*, 42(1):43–47, 1956.
- L. T. Tran. Kernel density estimation on random fields. *Journal of Multivariate Analysis*, 34(1):37–53, 1990.
- E. Valenzuela-Domínguez. *Asymptotic properties of nonparametric estimators for nonlinear functions of random fields*. PhD thesis, University of Kaiserslautern, 1995.